<h1 style="text-align:center">Completing the Square</h1>

Please note that the methods shown in the examples are not unique and if you find any errors, please let me know. This topic is A/A* at GCSE.

Express the following in the form $(x + a)^2 + b$, where a and b are constants.

Examples: 1. $x^2 + 4x + 5 = (x + 2)^2 - 2^2 + 5 = (x + 2)^2 - 4 + 5 = (x + 2)^2 + 1$

2. $x^2 - 6x + 5 = (x - 3)^2 - 3^2 + 5 = (x - 3)^2 - 9 + 5 = (x - 3)^2 - 4$

$-6 \div 2$ subtract $(-3)^2$

or simply subtract 3^2

Note: $(-3)^2$ is the same as 3^2

Try the following: (Grade A at GCSE)

1. $x^2 + 6x + 4$
2. $x^2 - 6x + 4$
3. $x^2 + 4x + 3$
4. $x^2 - 4x + 3$
5. $x^2 + 8x + 1$
6. $x^2 - 8x + 1$
7. $x^2 - 8x$

More challenging

Example: $x^2 + 3x + 4$

$$= \left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4$$

$$= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 4$$

$$= (x + \frac{3}{2})^2 + \frac{7}{4}$$

Try the following:

8. $x^2 + 3x - 4$
9. $x^2 - 3x + 4$
10. $x^2 + 5x + 1$
11. $x^2 - 5x + 1$
12. $7 + 5x + x^2$

Even More challenging

Questions with a negative coefficient of x^2

Grade A* at GCSE.

Example: Express $3 - 4x - x^2$ $in\ the\ form\ p - (x + r)^2$ where p and r are constants.

Solution: $3 - 4x - x^2 = -x^2 - 4x + 3$

$$= -(x^2 + 4x) + 3$$

$$= -\ ((x + 2)^2 - 2^2) + 3$$

$$= -((x + 2)^2 - 4) + 3$$

$$= -(x + 2)^2 + 4 + 3$$

$$= 7 - (x + 2)^2 \quad (p = 7, r = 2)$$

Try the following:

1. $1 + 4x - x^2$
2. $5 - 6x - x^2$
3. $2 + 8x - x^2$
4. $1 - 4x - x^2$
5. $1 + x - x^2$
6. $1 - 8x - x^2$

Additional Questions:

1. $2x^2 + 8x + 4$
2. $2x^2 - 8x + 1$
3. $2x^2 + 2x + 4$
4. $3 + 4x - 2x^2$
5. $5 + 8x - 2x^2$

Completing the square helps us to solve quadratic equations (where the quadratic does not factorise, although we may also use the formula) as well as to sketch the quadratic function.

It tells us about the minimum or maximum points and by looking at the transformation of the graph of

$y = f(x) = x^2$, we can easily sketch the quadratic.

Here is an example:

Typical Examination Question:

(a) Express $x^2 + 6x + 10$ $in\ the\ form\ (x + a)^2 + b\ where\ a\ and\ b\ are\ constants\ to\ be\ found.$
(b) Hence write down the minimum value of $x^2 + 6x + 10$.
(c) Hence write down the value x that gives this minimum value of $x^2 + 6x + 10$.
(d) Sketch the graph of
$y = x^2 + 6x + 10$ $showing\ the\ the\ intersction\ with\ the\ y-axis\ and\ the\ minimum\ point.$

Solution:

(a) $x^2 + 6x + 10$

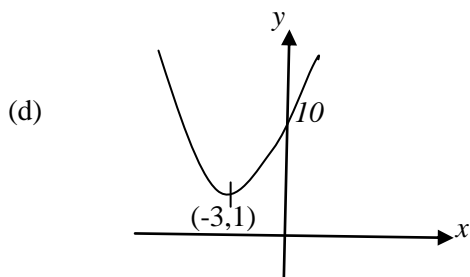 $= (x+3)^2 - 9 + 10 = (x+3)^2 + 1.$ $a = 3,\ b = 1.$

(b) Minimum value of $x^2 + 6x + 10$ *means the minimum* $y - value\ which\ is$ 1.

(c) $x = -3$ This comes from $(x+3) = 0$

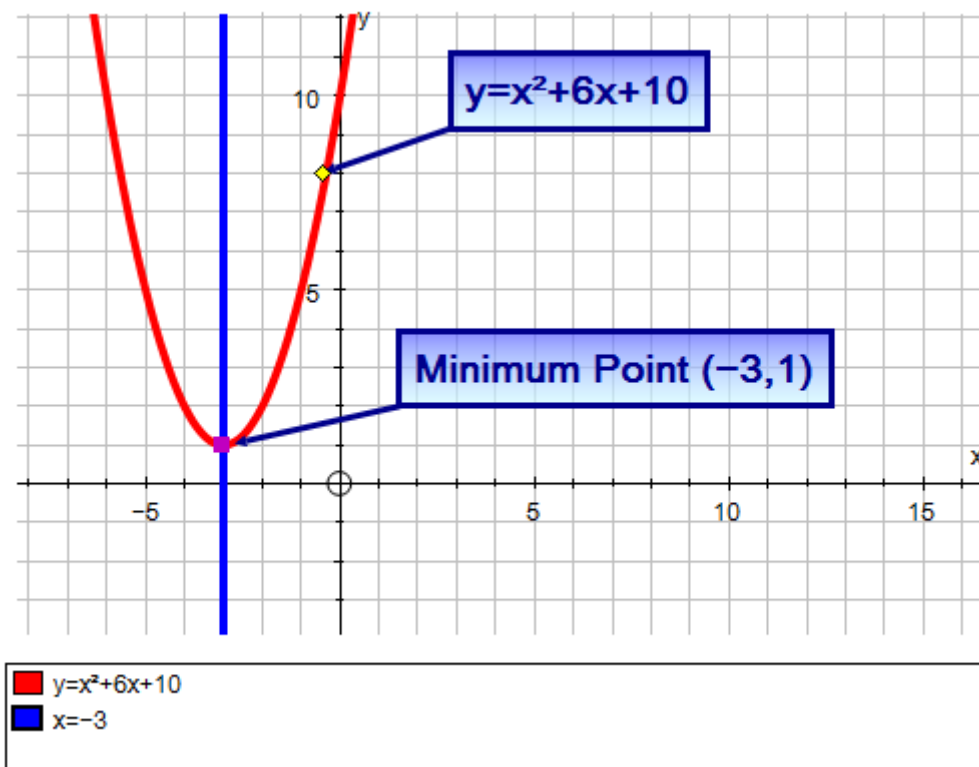NOTE: the line of symmetry is $x = -3$

Also $y = f(x) = x^2$ *has been transformed by a translation vector* $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

 to get $y = f(x+3) + 1 = (x+3)^2 + 1$

(d)

Using Autograph

Perfect Squares:

Examples:
1. $x^2 - 6x + 9 = (x - 3)^2$
2. $x^2 + 6x + 9 = (x + 3)^2$
3. $x^2 + 10x + 25 = (x + 5)^2$
4. $x^2 - 12x + 36 = (x - 6)^2$
5. $x^2 + 2x + 1 = (x + 1)^2$
6. $x^2 - 8x + 16 = (x - 4)^2$

Very special ones:

7. $4x^2 - 12x + 9 = (2x - 3)^2$ Note: *The coefficient of x^2 is* 4. $\sqrt{4} = 2$, also $\sqrt{9} = 3$

and 12 = 2x3 doubled.

The question may ask for the form $(ax + b)^2 + c$

8. $4x^2 - 12x + 10 = 4x^2 - 12x + 9 + 1 = (2x - 3)^2 + 1$
9. $9x^2 - 12x + 11 = 9x^2 - 12x + 4 + 7 = (3x - 2)^2 + 7$

Try these:
10. $25x^2 - 20x + 9$
11. $9x^2 + 24x + 19$
12. $16x^2 - 24x + 11$

Answers to questions on page1:

1. $(x + 3)^2 - 5$ 2. $(x - 3)^2 - 5$ 3. $(x + 2)^2 - 1$ 4. $(x - 2)^2 - 1$

5. $(x + 4)^2 - 15$ 6. $(x - 4)^2 - 15$ 7. $(x - 4)^2 - 16$

8. $\left(x + \dfrac{3}{2}\right)^2 - \dfrac{25}{4}$ 9. $\left(x - \dfrac{3}{2}\right)^2 - \dfrac{25}{4}$ 10. $\left(x + \dfrac{5}{2}\right)^2 - \dfrac{21}{4}$ 11. $\left(x - \dfrac{5}{2}\right)^2 - \dfrac{21}{4}$

12. $(x + \dfrac{5}{2})^2 + \dfrac{3}{4}$

Answers to questions on page2:

"Try the following"

1. $5 - (x - 2)^2$ 2. $14 - (x + 3)^2$ 3. $17 - (x - 4)^2$ 4. $5 - (x + 2)^2$

5. $\dfrac{5}{4} - (x - \dfrac{1}{2})^2$ 6. $17 - (x + 4)^2$

"Additional Questions"

1. $2(x + 2)^2 - 4$ 2. $2(x - 2)^2 - 7$ 3. $2\left(x + \dfrac{1}{2}\right)^2 - \dfrac{7}{2}$

4. $6 - 2(x - 1)^2$ 5. $13 - 2(x - 2)^2$

Answers to "Try these" from page 4

10. $(5x - 2)^2 + 5$ 11. $(3x + 4)^2 + 3$ 12. $(4x - 3)^2 + 2$

I have no doubt that there may be some errors! Please check all answers.

I hope you found this helpful.